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STUDY ON CONJUGATE HEAT TRANSFER BY VECTORIAL DIMENSIONAL ANALYSIS

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Abstract—Heat transfer in a system where two phases are conjugated is discussed from a somewhat general viewpoint using vectorial dimensional analysis which distinguishes the dimensions of length by phases as well as by spatial directions.

As a result, the following conclusions are derived: (1) To grasp conjugate heat transfer in a unified way, it is necessary to define new dimensionless groups which include conventional dimensionless groups defined in both phases. (2) In unsteady conjugate heat transfer there exists a new dimensionless group which represents an effect of the combination of physical properties of both phases.

NOMENCLATURE

a, thermal diffusivity;
b, coordinate normal to the paper;
B, physical properties parameter, $\lambda_f \rho_f c_f / \lambda_s \rho_s c_s$;
c, specific heat;
F, dimensionless functional relation;
$\boldsymbol{g}$, gravitational acceleration;
h, local heat-transfer coefficient;
h_m, mean heat-transfer coefficient;
H, fundamental dimension for heat;
l_x, l_y, l_z, characteristic length along x, y, z;
L_x, L_y, L_z, L_b, fundamental dimension for x-, y-, z-, b-coordinate;
M, fundamental dimension for mass;
Nu, Nusselt number, hx/λ_f;
Pe, Peclet number, $u_0 x/a_f$;
Pr_f, Prandtl number, ν_f/a_f;
q, heat flux;
q_0, intensity of plane heat source;
t, temperature;
T, fundamental dimension for time;
u, velocity;
u_0, reference velocity;
x, y, z, coordinate.

Greek symbols

β, coefficient of thermal expansion;
θ, temperature difference from some reference temperature;
Θ, fundamental dimension for temperature;
λ, thermal conductivity;
μ, dynamic viscosity;
ν, kinematic viscosity;
ρ, density;
τ, time.

Subscripts

f, fluid or phase f;
s, solid or phase s;
w, surface.

1. INTRODUCTION

WHEN one tries to arrange heat transfer in a system where two phases are thermally conjugated (conjugate heat transfer) in a dimensionless form, one is often encountered with difficulties in choosing dimensionless groups, because conventional dimensionless groups are defined in a single phase. For example, a dimensionless group for a local heat-transfer coefficient is the Nusselt number from the fluid side, but the Biot number should be considered instead from the solid side. Thus, two conventional dimensionless groups exist for one physical parameter. The situation is the same with dimensionless groups for time, coordinate along the boundary surface etc.

Previous research on conjugate heat transfer [1–6] indicates that it is not sufficient to use conventional dimensionless groups defined in a single phase, and it is necessary to use newly defined dimensionless groups which are composed of conventional dimensionless groups and physical properties, characteristic lengths of both phases, to arrange conjugate heat-transfer problems in dimensionless forms. However these studies are rather apt to confine themselves to their specific problems and it seems that trials to see conjugate heat transfer in a unified way are few.

In this paper, conjugate heat transfer is studied from the viewpoint that both phases are on an equal footing, and discussions are made to seek a new way of defining dimensionless groups which will enable us to see conjugate heat transfer in a unified way, by vectorial dimensional analysis in which the dimensions of length are distinguished not only by their spatial directions but also by phases.

2. METHOD OF VECTORIAL DIMENSIONAL ANALYSIS FOR CONJUGATE TRANSFER PHENOMENA

In this section, the method of vectorial dimensional analysis in which dimensions of length are distinguished by phases is described.

The method of ordinary vectorial dimensional analysis [7] distinguishes the dimensions of length only by their spatial directions. However, in conjugate transfer phenomena, where fluxes go through the boundary surface from one phase to the other phase of different physical properties, one must further distinguish the axis of the coordinate normal to the boundary surface by phases into two axes on the occasion of vectorial dimensional analysis. In Fig. 1, phase f and phase s are conjugated by the surface heat

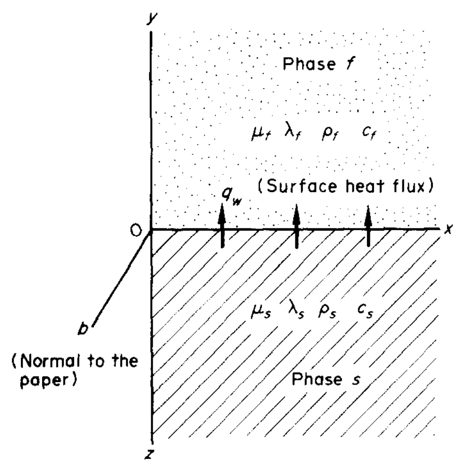

FIG. 1. Schematic representation of the conjugate transfer problem.

flux q_w. Then the x-coordinate is taken along the boundary surface (the fundamental dimension of length along this direction is taken to be L_x), the b-coordinate normal to the paper (L_b), the y-coordinate normal to the boundary surface in phase $f(L_y)$, and the z-coordinate normal to the boundary surface in phase $s(L_z)$. Phase f and phase s possess in common the x-coordinate and the b-coordinate at the boundary surface, so x and b need not to be distinguished by phases.

In the following, an application example of the method of vectorial dimensional analysis which distinguishes the dimensions of length by phases is presented.

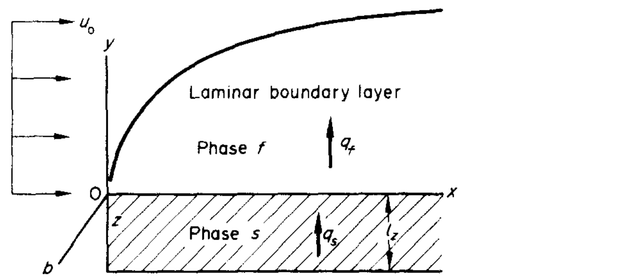

FIG. 2. Application example of vectorial dimensional analysis for conjugate transfer phenomena.

In Fig. 2, let phase f be a parallel flow with velocity u_0 and phase s be a half-infinite flat plate with thickness l_z. At time $\tau < 0$, a laminar boundary layer flow is formed on the flat plate from the leading edge of the plate and the plate has a uniform temperature θ_0 (temperatures are measured from the level of the bulk temperature of the fluid). Then at time $\tau = 0$, the device to keep the temperature of the plate constant is cut off and the plate is left to be cooled by the flow. It is required to seek the transient characteristics of the surface temperature of the plate $\theta_w = \theta_w(\tau, x)$, the temperature distribution in the fluid $\theta_f = \theta_f(\tau, x, y)$, the temperature distribution in the solid $\theta_s = \theta_s(\tau, x, z)$ and the local heat-transfer coefficient $h = h(\tau, x)$.

Physical quantities concerning this two-dimensional unsteady conjugate heat transfer and their dimensions are listed in Table 1. The predominant directions of the heat fluxes are estimated to be parallel to y in the fluid, and z in the solid. Moreover, a product of the density and the specific heat is treated as one physical quantity in the solid.

The dimensionless functional relation for the surface temperature is derived as follows, by treating θ_w, τ, x, λ_s and μ_f as independent physical quantities:

$$\frac{\theta_w}{\theta_0} = F(T_{fs}, X_{fs}, B, Pr_f) \tag{1}$$

where

$$\left.\begin{aligned} T_{fs} &= \frac{\tau\lambda_f\rho_f c_f}{l_z^2(\rho_s c_s)^2} \\ X_{fs} &= \frac{x\lambda_f\rho_f c_f}{u_0 l_z^2(\rho_s c_s)^2} \\ B &= \frac{\lambda_f\rho_f c_f}{\lambda_s\rho_s c_s} \end{aligned}\right\} \tag{2}$$

and Pr_f is the Prandtl number of the fluid.

Table 1. Dimensions of physical quantities

	h	θ_w	θ_f	θ_s	τ	x	y	z	λ_s	μ_f	λ_f	ρ_f	c_f	$\rho_s c_s$	l_z	u_0	θ_0
L_x	−1	0	0	0	0	1	0	0	−1	−1	−1	−1	0	−1	0	1	0
L_y	0	0	0	0	0	0	1	0	0	1	1	−1	0	0	0	0	0
L_z	0	0	0	0	0	0	0	1	1	0	0	0	0	−1	1	0	0
M/L_b	0	0	0	0	0	0	0	0	0	1	0	1	−1	0	0	0	0
H/L_b	1	0	0	0	0	0	0	0	1	0	1	0	1	1	0	0	0
Θ	−1	1	1	1	0	0	0	0	−1	0	−1	0	−1	−1	0	0	1
T	−1	0	0	0	1	0	0	0	−1	−1	−1	0	0	0	0	−1	0

The dimensionless functional relations for the temperature distributions in the fluid and in the solid are obtained by treating θ_f or θ_s, τ, x, y or z, λ_s and μ_f as independent physical quantities:

$$\frac{\theta_f}{\theta_0} = F\left(T_{fs},\ X_{fs},\ \frac{y\rho_f c_f}{l_z \rho_s c_s},\ B,\ Pr_f\right) \quad (3)$$

$$\frac{\theta_s}{\theta_0} = F\left(T_{fs},\ X_{fs},\ \frac{z}{l_z},\ B,\ Pr_f\right). \quad (4)$$

The dimensionless functional relation for the local heat-transfer coefficient is also obtained by treating h, τ, x, λ_s and μ_f as independent physical quantities:

$$H_{fs} = F(T_{fs},\ X_{fs},\ B,\ Pr_f) \quad (5)$$

where

$$H_{fs} = \frac{h l_z \rho_s c_s}{\lambda_f \rho_f c_f}. \quad (6)$$

As to the derivation procedure of the above relations, see Appendix.

For comparison, the result obtained by ordinary dimensional analysis (in which directions and phases are not distinguished) and that by ordinary vectorial dimensional analysis (in which phases are not distinguished) for the local heat-transfer coefficient are stated:

$$Nu = F\left(\frac{u_0\tau}{x},\ \frac{x}{l_z},\ \frac{u_0 x}{\nu_f}\ \frac{\lambda_f}{\lambda_s}\ \frac{\rho_f}{\rho_s}\ \frac{c_f}{c_s}\ Pr_f\right) \quad (7)$$

in ordinary dimensional analysis, and

$$\frac{Nu}{Pe^{\frac{1}{2}}} = F\left(\frac{u_0\tau}{x},\ \frac{a_f x}{u_0 l_z^2},\ \frac{\lambda_f}{\lambda_s},\ \frac{\rho_f}{\rho_s},\ \frac{c_f}{c_s},\ Pr_f\right) \quad (8)$$

in ordinary vectorial dimensional analysis.

On the other hand, equation (5) can be rewritten using relations

$$(H_{fs})^1 \cdot (X_{fs})^{\frac{1}{2}} = \frac{Nu}{Pe^{\frac{1}{2}}}$$

$$(T_{fs})^1 \cdot (X_{fs})^{-1} = \frac{u_0\tau}{x}$$

$$(X_{fs})^1 \cdot (B)^{-1} = \frac{a_s x}{u_0 l_z^2}$$

into

$$\frac{Nu}{Pe^{\frac{1}{2}}} = F\left(\frac{u_0\tau}{x},\ \frac{a_s x}{u_0 l_z^2},\ B,\ Pr_f\right). \quad (9)$$

Comparing equation (9) with equations (7) and (8), it is clear that the vectorial dimensional analysis method which distinguishes phases gives dimensionless functional relations more compact than ordinary methods do.

3. DIMENSIONLESS GROUPS FOR CONJUGATE HEAT TRANSFER

Dimensionless groups obtained by applying the method described in the preceding section to some conjugate heat transfer problems are tabulated in Table 2 along with corresponding conventional dimensionless groups. As clearly seen from the table, dimensionless groups induced by vectorial dimensional analysis for conjugate transfer phenomena are defined using characteristic lengths and physical properties of both phases (in the following, these will be called 'conjugate dimensionless groups'), whereas conventional dimensionless groups are defined using characteristic lengths and physical properties of only one phase. In Table 2, only common representative conjugate dimensionless groups are shown. Authors do not assert that all the conjugate dimensionless groups should be used instead of conventional dimensionless groups to arrange any specific conjugate problem. Authors can say that these conjugate dimensionless groups enable us to grasp conjugate problems general in a unified way as will be shown in the next section, and in some problems it is indispensable to use conjugate dimensionless groups to arrange the results completely.

It must be also noted that a parameter defined by $\lambda_f \rho_f c_f/\lambda_s \rho_s c_s$ appears in unsteady conjugate problems due to the existence of heat conduction in phase s as well as in phase f. This parameter, which will be called 'physical properties parameter', represents the effect of the combination of physical properties of both phases (fluid–solid, solid–solid) on unsteady conjugate heat transfer, and will be discussed in the next section.

4. INTERPRETATIONS OF PREVIOUSLY STUDIED CONJUGATE HEAT-TRANSFER PROBLEMS

In this section, trials are made to interpret dimensionless groups variously defined in previous researches on conjugate heat transfer in a unified way using new dimensionless groups tabulated in Table 2.

At this stage it must be convenient to point out that there exists a definite relation between the forms of conjugate dimensionless groups and those of conventional dimensionless groups. It is easy to check the relation.

(conjugate dimensionless group) = (conventional dimensionless group defined in the fluid)$\cdot\left(\frac{x\rho_f c_f}{l_z \rho_s c_s}\right)^m$

or

(conjugate dimensionless group) = (conventional dimensionless group defined in the solid)$\cdot\left(\frac{\lambda_f \rho_f c_f}{\lambda_s \rho_s c_s}\right)^n$

where m and n are appropriate indices and when the characteristic length l_x exists, x in $x\rho_f c_f/l_z \rho_s c_s$ may be replaced by l_x. For convenience, $x\rho_f c_f/l_z \rho_s c_s$ and $\lambda_f \rho_f c_f/\lambda_s \rho_s c_s$ will be called 'bridging multiplier'. In the following discussions this bridging multiplier makes it easier to rewrite dimensionless groups.

Unsteady problems are discussed to begin with.

First, a classical contact problem as shown in Fig. 3 [8] is taken up. A solid (phase f) with uniform temperature t_{f0} and a solid (phase s) with uniform temperature t_{s0} are suddenly put into contact at time $\tau = 0$. Transient characteristics of temperature distributions in both phases t_f, t_s and the boundary surface temperature t_w are to be studied.

Variables to specify the situation are τ, y, z, $\theta_0(=t_{s0}-t_{f0})$, λ_f, $\rho_f c_f$, λ_s, $\rho_s c_s$ and unknown variables

Table 2. Dimensionless groups

Physical parameter	Conjugate	Defined in the fluid	Defined in the solid	Remarks
h	$H_{fs} \equiv \frac{hl_z\rho_s c_s}{\lambda_f\rho_f c_f}$	$\frac{hx}{\lambda_f}$	$\frac{hl_z}{\lambda_s}$	
τ	$T_{fs} \equiv \frac{\tau\lambda_f\rho_f c_f}{l_z^2(\rho_s c_s)^2}$	$\frac{a_f\tau}{x^2}$	$\frac{a_s\tau}{l_z^2}$	
x	$X_{fs} \equiv \frac{x\lambda_f\rho_f c_f}{u_0 l_z^2(\rho_s c_s)^2}$	$\frac{a_f}{u_0 x}$	$\frac{a_s x}{u_0 l_z^2}$	forced convection
θ	$\Theta^*_{fs} \equiv \frac{g\beta_f\theta l_z^4(\rho_s c_s)^4}{x(\lambda_f\rho_f c_f)^2}$ †	$\frac{g\beta_f\theta x^3}{a_f^2}$ †	—	natural convection
	$\Theta_{fs} \equiv \frac{1}{Pr_f^2}\cdot\Theta^*_{fs}$ †	$\frac{g\beta_f\theta x^3}{\nu_f^2}$ †	—	
q_0	$Q^*_{fs} \equiv \frac{g\beta_f q_0 l_z^5(\rho_s c_s)^5}{x(\lambda_f\rho_f c_f)^3}$ †	$\frac{g\beta_f q_0 x^4}{\lambda_f a_f^2}$ †	—	natural convection
	$Q_{fs} \equiv \frac{1}{Pr_f^2}\cdot Q^*_{fs}$ †	$\frac{g\beta_f q_0 x^4}{\lambda_f \nu_f^2}$ †	—	
θ or q_0	$R_{fs} \equiv \frac{\theta\lambda_f\rho_f c_f}{q_0 l_z\rho_s c_s}$	—	—	forced convection with plane heat source
combination effect	$B \equiv \frac{\lambda_f\rho_f c_f}{\lambda_s\rho_s c_s}$	—	—	unsteady problem

† When characteristic length l_x exists, x may be replaced by l_x.

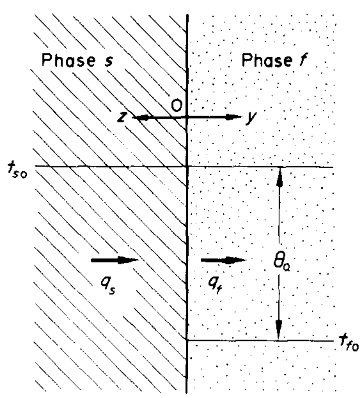

FIG. 3. One-dimensional contact problem.

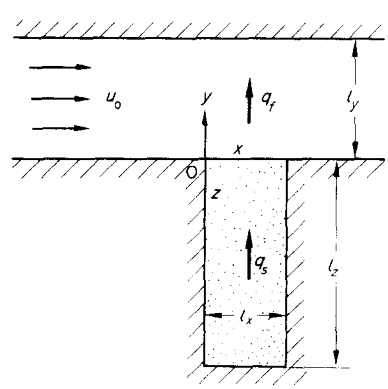

FIG. 4. Luikov–Perel'man problem.

are $\theta_w(=t_w-t_{f0})$, $\theta_f(=t_f-t_{f0})$ and $\theta_s(=t_s-t_{f0})$. Then the following dimensionless functional relations are obtained applying the method described in Section 2:

$$\frac{\theta_w}{\theta_0} = F(B) \tag{10}$$

$$\frac{\theta_f}{\theta_0} = F\left(\frac{y}{\sqrt{(a_f\tau)}}, B\right) \tag{11}$$

$$\frac{\theta_s}{\theta_0} = F\left(\frac{z}{\sqrt{(a_s\tau)}}, B\right). \tag{12}$$

Analytical solutions for this problem are given as (rearranging in terms of the physical properties parameter B)

$$\frac{\theta_w}{\theta_0} = \frac{1}{1+\sqrt{B}} \tag{13}$$

$$\frac{\theta_f}{\theta_0} = \frac{1}{1+\sqrt{B}}\cdot\text{erfc}\left(\frac{y}{2\sqrt{(a_f\tau)}}\right) \tag{14}$$

$$\frac{\theta_s}{\theta_0} = \frac{1}{1+\sqrt{B}}\cdot\left\{1+\sqrt{B}\cdot\text{erf}\left(\frac{z}{2\sqrt{(a_s\tau)}}\right)\right\}. \tag{15}$$

These solutions are substantially equal to the results, equations (10)–(12).

Next the unsteady problem studied by Luikov and

Perel'man [9] is taken up. As shown in Fig. 4, at one wall of a two-dimensional channel (width l_y and initial temperature t_0) through which flows a completely developed flow with temperature t_∞, is placed a solid body of dimensions $l_x \cdot l_z$ whose entire surface except the surface contacting with the fluid is thermally insulated. Luikov and Perel'man treated this problem (the transient characteristics of the mean surface temperature and the mean heat-transfer coefficient for length l_x) introducing an effective value of velocity u_0 which considers not only the velocity boundary layer but also the thermal boundary layer of the flow. Defining the dimensionless groups

$$K = B^{\frac{1}{2}}, \quad \Lambda = \left(\frac{u_0 l_z^2}{a_s l_x}\right)^{\frac{1}{2}}, \quad \xi = \frac{u_0 \tau}{l_x} \tag{16}$$

they obtained the following dimensionless functional relation for the mean heat-transfer coefficient:

$$\frac{h_m l_z}{\lambda_s} = F(K, \Lambda, \xi). \tag{17}$$

The vectorial dimensional analysis method for conjugate transfer phenomena gives the following dimensionless functional relation for the mean heat-transfer coefficient:

$$H_{fsm} = F(T_{fs}, B, X_{fsl}) \tag{18}$$

where

$$H_{fsm} = \frac{h_m l_z \rho_s c_s}{\lambda_f \rho_f c_f}, \quad X_{fsl} = \frac{l_x \lambda_f \rho_f c_f}{u_0 l_z^2 (\rho_s c_s)^2}.$$

Equation (18) can be rewritten into

$$\frac{h_m l_z}{\lambda_s} = F(K, \Lambda, \xi) \tag{19}$$

and this is equal to the result given by Luikov and Perel'man.

Next the unsteady problem studied by Adams and Gebhert [10] is taken up. As shown in Fig. 5, a two-dimensional plate of dimensions $l_x \cdot l_z$, on which surface there is a plane heat source with intensity q_0, is placed in a two-dimensional uniform flow with velocity u_0 and temperature t_∞. Adams and Gebhart showed that the transient surface temperature $\theta_w (= t_w - t_\infty)$, after putting the plane heat source into run step-wise at time $\tau = 0$, falls between the two solutions which are obtained by solving the solid side energy equation (including only heat capacity of the solid) under the assumption that the surface temperature is constant in one case, with the other the surface heat flux being kept constant. They gave the following expression:

$$\frac{\theta_w}{\theta_{w\infty}} = 1 - \exp\left(-\frac{\tau'}{Q}\right) \tag{20}$$

where

$$\tau' = \frac{u_0 \tau}{l_x}, \quad Q = \frac{l_z \rho_s c_s u_0 \theta_{w\infty}}{l_x q_0}.$$

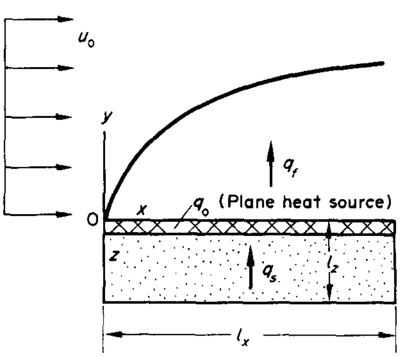

FIG. 5. Adams–Gebhart problem.

By vectorial dimensional analysis for conjugate transfer phenomena, the following dimensionless functional relation for the surface temperature is obtained:

$$R_{fsw} = \frac{\theta_w \lambda_f \rho_f c_f}{q_0 l_z \rho_s c_s} = F(T_{fs}, X_{fs}, B, Pr_f). \tag{21}$$

When the parameter B is nearly zero as Adams and Gebhart treated, and the final surface temperature $\theta_{w\infty}$ depends on q_0, the l.h.s. of equation (21) stands for $\theta_w/\theta_{w\infty}$, then equation (21) can be rewritten using dimensionless groups defined by Adams and Gebhart:

$$\frac{\theta_w}{\theta_{w\infty}} = F\left(\frac{\tau'}{Q}, X_{fs}, Pr_f\right). \tag{22}$$

Thus, equation (22) is substantially equal to the result given by Adams and Gebhart.

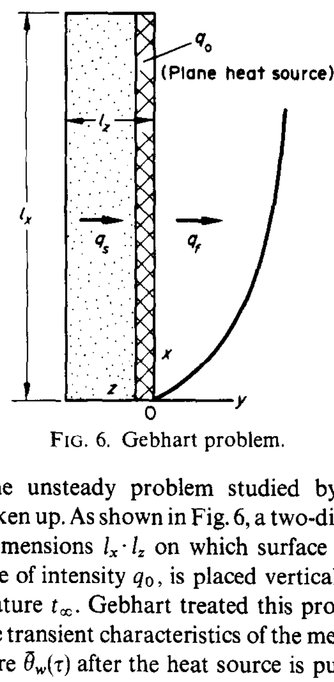

FIG. 6. Gebhart problem.

Next the unsteady problem studied by Gebhart [1–5] is taken up. As shown in Fig. 6, a two-dimensional plate of dimensions $l_x \cdot l_z$ on which surface is a plane heat source of intensity q_0, is placed vertically in fluid of temperature t_∞. Gebhart treated this problem concerning the transient characteristics of the mean surface temperature $\bar{\theta}_w(\tau)$ after the heat source is put into run at time $\tau = 0$ and obtained the following result:

$$\frac{\bar{\theta}_w}{\bar{\theta}_{w\infty}} = F(T_G, Q_G, Pr_f) \tag{23}$$

where

$$\left.\begin{aligned} T_G &= \left(\frac{a_f \tau}{l_x^2}\right) \cdot (b \cdot G_G \cdot Pr_f)^{2/5} \\ Q_G &= \left(\frac{l_z \rho_s c_s}{l_x \rho_f c_f M}\right) \cdot (b \cdot G_G \cdot Pr_f)^{1/5} \\ G_G &= \frac{g \beta_f l_x^4 q_0}{\lambda_f \nu_f^2} \end{aligned}\right\} \tag{24}$$

and b, M are both functions of Pr_f only.

Applying vectorial dimensional analysis for conjugate transfer phenomena the following dimensionless functional relation is obtained for the mean surface temperature:

$$\bar{R}_{fsw} = \frac{\bar{\theta}_w \lambda_f \rho_f c_f}{q_0 l_z \rho_s c_s} = F(T_{fs},\ Q^*_{fs},\ B,\ Pr_f). \qquad (25)$$

The arrangement of this problem by Gebhart is in itself excellent and the definitions of the dimensionless time T_G and the dimensionless heat capacity of the plate Q_G mean that the dimensionless intensity of the plane heat source serves as if it were the bridging multiplier. When one stands on the point of view that characteristic lengths and physical properties are additionally used to construct conjugate dimensionless groups, one is to select the dimensionless time and the dimensionless intensity of the plane heat source as the physical parameters. Actually, the following relations between the definitions by Gebhart and the conjugate dimensionless groups show that equations (23) and (25) are substantially equal:

$$T_{fs} = \frac{a_f \tau}{l_x^2} \cdot \left(\frac{l_x \rho_f c_f}{l_z \rho_s c_s}\right)^2 \sim (T_G)^1 \cdot (Q_G)^{-2}$$

$$Q^*_{fs} = Pr_f^{\frac{2}{3}} \cdot G_G \cdot \left(\frac{l_z \rho_s c_s}{l_x \rho_f c_f}\right)^5 \sim (Q_G)^5.$$

From above discussions on unsteady problems, it is easily seen that the effect of the combination of physical properties of both phases is represented only by the physical properties parameter B, and this is originated from taking account of heat conduction in the solid. However, B may not have so great an effect on the problems with heat sources as on those without. The problem studied by Gebhart is considered to be this case.

However, in conjugate problems between a moving fluid (phase f) and a moving fluid (phase s) the parameter B will not appear, and there will appear two parameters $\lambda_f \rho_f / \lambda_s \rho_s$ and c_f / c_s instead of B, because in the moving fluid the density and the specific heat work independently.

Now steady problems are discussed in the followings. In steady problems the heat capacity in a unit volume of the solid phase $\rho_s c_s$ does not have any effect on the phenomena. Therefore, in steady problems, the physical properties parameter B does not appear and dimensionless groups take irregular forms, different from those tabulated in Table 2.

First as a steady problem, the one studied by Lock and Gunn [6] is taken up. As shown in Fig. 7, a two-dimensional slender fin of root width $2 \cdot l_z$ and length l_x is placed downward in a fluid with temperature t_∞. Lock and Gunn treated this problem concerning the steady state temperature distribution at the fin surface when the fin root temperature is maintained at a constant temperature $t_R (\theta_R = t_R - t_\infty)$ under the conditions that the fin slenderness ratio l_x / l_z is sufficiently large and that the half width of the fin $\delta(x)$ at an arbitrary position x is given by

$$\frac{\delta}{l_z} = \left(\frac{x}{l_x}\right)^m. \qquad (26)$$

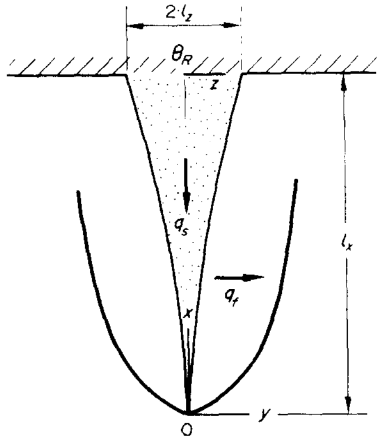

FIG. 7. Lock–Gunn problem.

They concluded that the surface temperature expressed (or approximated) by the form

$$\frac{\theta_w}{\theta_R} = \left(\frac{x}{l_x}\right)^n \qquad (27)$$

is determined by the parameter

$$\chi = \frac{\lambda_f l_x}{\lambda_s l_z} \cdot \left\{\frac{g \beta_f \theta_R l_x^3}{a_f^2 (1 + Pr_f)}\right\}^{\frac{1}{4}} \qquad (28)$$

only.

In this problem, the predominant direction of the heat flux in the fluid q_f is y-direction and the predominant direction of the heat flux in the solid q_s is supposed to be x-direction, then the vectorial dimensional analysis method for conjugate transfer phenomena gives the following dimensionless functional relation for the surface temperature:

$$\frac{\theta_w}{\theta_R} = F\left\{\frac{x}{l_x},\ \left(\frac{\lambda_f l_x}{\lambda_s l_z}\right)^4 \cdot \frac{g \beta_f \theta_R l_x^3}{a_f^2},\ Pr_f\right\}. \qquad (29)$$

This result is substantially equal to the result given by Lock and Gunn. If this problem is regarded to be an unsteady one, the dimensionless functional relation for the surface temperature is given by

$$\frac{\theta_w}{\theta_R} = F\left(T_{fs},\ \frac{x}{l_x},\ \frac{l_x^2 \lambda_f \rho_f c_f}{l_z^2 \lambda_s \rho_s c_s},\ \Theta^*_{fs},\ Pr_f\right). \qquad (30)$$

In this relation, note that the physical properties parameter B does not appear in spite of the unsteady problem, since the predominant direction of q_s is parallel to the x-coordinate. When one treats this problem as a steady one, two parameters will disappear, one being T_{fs}, and the other the third dimensionless group on the right-hand side of equation (30). By this time, the fourth dimensionless group which includes $\rho_s c_s$ takes the form

$$(\Theta^*_{fs})^1 \cdot \left(\frac{l_x^2 \lambda_f \rho_f c_f}{l_z^2 \lambda_s \rho_s c_s}\right)^4 = \left(\frac{\lambda_f l_x}{\lambda_s l_z}\right)^4 \cdot \frac{g \beta_f \theta_R l_x^3}{a_f^2} \qquad (31)$$

and this may be regarded as an irregular form of Θ^*_{fs}.

Next the steady problem studied by Zinnes [1] is taken up. As shown in Fig. 8, a two-dimensional solid of dimensions $l_x \cdot l_z$ whose entire surface, but for the part of which contacting with a fluid of temperature t_∞, is thermally insulated, is placed on a two-dimensional vertical wall. On the upper and lower parts of the

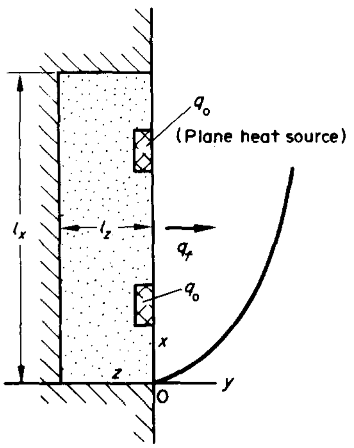

FIG. 8. Zinnes problem.

surface of the solid contacting with the fluid are placed plane heat sources of intensity q_0. Zinnes treated this problem concerning steady state natural convection heat transfer after heat sources being put into run, and obtained following relations for the surface temperature and the temperature distribution in the fluid:

$$\frac{g\beta_f\theta_w l_x^3}{v_f^2} = F\left(X,\ G_z,\ \frac{\lambda_f}{\lambda_s},\ Pr_f\right) \tag{32}$$

$$\frac{g\beta_f\theta_f l_x^3}{v_f^2} = F\left(X,\ Y,\ G_z,\ \frac{\lambda_f}{\lambda_s},\ Pr_f\right) \tag{33}$$

where

$$X = \frac{x}{l_x},\ Y = \frac{y}{l_x},\ G_z = \frac{g\beta_f q_0 l_x^4}{\lambda_f v_f^2}. \tag{34}$$

In this problem, the predominant direction of q_f is y-direction, on the other hand that of q_s is expected to be much more complicated according to the running way of plane heat sources and/or the region considered, so two cases, the case $q_s//z$ and the case $q_s//x$, are taken account of. Then the following results are obtained for the surface temperature by vectorial dimensional analysis for conjugate transfer phenomena:

$$\left(\frac{\lambda_f l_z}{\lambda_s l_x}\right)^4 \cdot \frac{g\beta_f\theta_w l_x^3}{a_f^2} = F\left\{\frac{x}{l_x},\ \left(\frac{\lambda_f l_z}{\lambda_s l_x}\right)^5 \cdot \frac{g\beta_f q_0 l_x^4}{\lambda_f a_f^2},\ Pr_f\right\} \tag{35}$$

for the case $q_s//z$, and

$$\left(\frac{\lambda_f l_x}{\lambda_s l_z}\right)^4 \cdot \frac{g\beta_f\theta_w l_x^3}{a_f^2} = F\left\{\frac{x}{l_x},\ \left(\frac{\lambda_f l_x}{\lambda_s l_z}\right)^5 \cdot \frac{g\beta_f q_0 l_x^4}{\lambda_f a_f^2},\ Pr_f\right\} \tag{36}$$

for the case $q_s//x$.

Comparison of equations (35) or (36) with equation (32) given by Zinnes indicates that they are substantially equal and furthermore that the result by Zinnes becomes more compact expression by using dimensionless groups which appear in equations (35) or (36). This means that the parameter λ_f/λ_s does not work independently. If this problem is regarded as an unsteady one, the dimensionless functional relation for the surface temperature is given by (for simplicity only the case $q_s//z$ is considered)

$$\Theta_{fs}^* = F\left(T_{fs},\ \frac{x}{l_x},\ Q_{fs}^*,\ B,\ Pr_f\right). \tag{37}$$

Therefore, in the steady problem, T_{fs} and B will disappear, the latter affecting the forms of other dimensionless groups containing the term $\rho_s c_s$, as follows

$$(\Theta_{fs}^*)^1 \cdot (B)^4 = \left(\frac{\lambda_f l_z}{\lambda_s l_x}\right)^4 \cdot \frac{g\beta_f\theta_w l_x^3}{a_f^2} \tag{38}$$

$$(Q_{fs}^*)^1 \cdot (B)^5 = \left(\frac{\lambda_f l_z}{\lambda_s l_x}\right)^5 \cdot \frac{g\beta_f q_0 l_x^4}{\lambda_f a_f^2}. \tag{39}$$

Thus, equation (38) may be regarded as another irregular form of Θ_{fs}^* and equation (39) as an irregular form of Q_{fs}^*.

5. CONCLUDING REMARKS

By introducing the method of vectorial dimensional analysis which distinguishes the dimensions of length by phases, a perspective view of conjugate heat transfer can be obtained. That is to say, to treat conjugate heat-transfer problems in a unified way it is necessary to use conjugate dimensionless groups, and in unsteady conjugate heat-transfer problems between a fluid (or solid) and a solid there exists the physical properties parameter B which expresses the effect of the combination of physical properties of the fluid (or solid) and of the solid on heat transfer.

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APPENDIX

The derivation procedure of equation (3), which is picked up as a representative from among equations (1), (3), (4) and (5), is described in this Appendix.

Physical quantities which affect the temperature distribution in the fluid θ_f are τ, x, y, μ_f, λ_f, ρ_f, c_f, λ_s, $\rho_s c_s$, l_z, u_0 and θ_0. First, the predominant directions of the heat flux in phase $f(q_f)$ and of that in phase $s(q_s)$ in the problem must be estimated. Then the dimensions of λ_f and λ_s can be determined by Fourier's law of heat conduction. In this example, q_f and q_s are estimated to be parallel to y and z respectively, so $[\lambda_f] = [q_f]/[\partial\theta_f/\partial y] = HL_y/L_x L_b \Theta T$ and $[\lambda_s] = [q_s]/[\partial\theta_s/\partial z] = HL_z/L_x L_b \Theta T$. After the determination of dimensions of all physical quantities concerned (see Table 1), independent physical quantities must be appropriately chosen. Since independent fundamental dimensions for this example are L_x, L_y, L_z, M/L_b, H/L_b, Θ and T, six physical quantities (θ_f, τ, x, y, λ_s, μ_f) are chosen as independent. Then Buckingham's Pi theorem gives the following functional relation:

$$\pi_1 = F(\pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$$

where

$$\pi_1 = \frac{\theta_f}{\theta_0}, \qquad \pi_2 = \frac{\tau\lambda_f\rho_f c_f}{l_z^2(\rho_s c_s)^2}, \qquad \pi_3 = \frac{x\lambda_f\rho_f c_f}{u_0 l_z^2(\rho_s c_s)^2},$$

$$\pi_4 = \frac{y\rho_f c_f}{l_z\rho_s c_s}, \qquad \pi_5 = \frac{\lambda_s\rho_s c_s}{\lambda_f\rho_f c_f}, \qquad \pi_6 = \frac{\mu_f c_f}{\lambda_f}.$$

For convenience, the form $\lambda_f\rho_f c_f/\lambda_s\rho_s c_s$ is used instead of π_5 in this paper.

ETUDE PAR L'ANALYSE DIMENSIONNELLE VECTORIELLE DU TRANSFERT DE CHALEUR CONJUGUE

Résumé—Le transfert de chaleur dans un système comprenant deux phases est étudié du point de vue assez général de l'analyse dimensionnelle vectorielle, en distinguant les dimensions de longueur suivant les phases aussi bien que suivant les directions spatiales.

Comme résultat, on a tiré les conclusions suivantes: (1) Afin de saisir le phénomène du transfert de chaleur conjugué dans un mélange de phases, de manière unifiée, il est nécessaire de définir de nouveaux groupements adimensionnels qui comprennent en outre les groupements adimensionnels habituels définis dans chacune des phases; (2) Dans le cas du transfert de chaleur instationnaire, il existe un nouveau groupement adimensionnel qui représente l'effet combiné des propriétés physiques de chaque phase.

UNTERSUCHUNG DES ZUSAMMENGESETZTEN WÄRMEÜBERGANGS DURCH VEKTORIELLE DIMENSIONSANALYSE

Zusammenfassung—Der Wärmeübergang in einem System mit zwei zusammengesetzten Phasen wird von einem allgemeineren Gesichtspunkt aus untersucht mit Hilfe der vektoriellen Dimensionsanalyse, wobei nach Dimensionen der Länge wie auch des Raumes unterschieden wird.

Das Ergebnis sind folgende Schlußfolgerungen: (1) Um den zusammengesetzten Wärmeübergang in einheitlicher Weise zu erfassen ist es notwendig, neue dimensionslose Gruppen zu definieren, die konventionelle dimensionslose Gruppen für beide Phasen umfassen; (2) Bei instationärem zusammengesetztem Wärmeübergang ergibt sich eine neue dimensionslose Gruppe, die den Einfluß der Kombination der physikalischen Stoffwerte beider Phasen wiedergibt.

ИССЛЕДОВАНИЕ СОПРЯЖЕННОГО ПРОЦЕССА ПЕРЕНОСА ТЕПЛА С ПОМОЩЬЮ ВЕКТОРНОГО АНАЛИЗА РАЗМЕРНОСТЕЙ

Аннотация — Обсуждается перенос тепла в системе из двух сопряженных фаз с несколько общей точки зрения с помощью векторного анализа размерностей, в котором учитываются различные масштабы длины как по фазам, так и по направлениям в пространстве.

В результате сделаны следующие выводы: (1) для унифицированного описания ппоцесса переноса тепла необходимо определить новые безразмерные комплексы, которые включают обычные безразмерные комплексы, определяемые в обеих фазах; (2) при нестационарном сопряженном переносе тепла существует новый безразмерный комплекс, который характеризует влияние сочетания физических характеристик обеих фаз.